SS 2016

Dr. Milena Pabiniak
Marc Kegel

Differential Geometry

Homework 10

Mandatory Exercise 1. (10 points)
Show that the curvature of $S^n \subset R^{n+1}$, with the metric induced from R^{n+1}, is constant.
Hint: $SO(n+1)$ acts transitively on S^n, by isometries, so it is enough to show that S^n is isotropic at $p = (1, 0, \dots, 0)$. Find the isotropy group $G_p \subset SO(n+1)$ of p and analyze the G_p action on $T_pS^n \cong R^n \subset T_pR^{n+1} \cong R^{n+1}$. Is it transitive on 2-planes?

Mandatory Exercise 2. (10 points)
Show that $||X_p||^2||Y_p||^2 - \langle X_p, Y_p\rangle^2$ gives us the square of the area of the parallelogram in T_pM spanned by X_p, Y_p. Conclude that the sectional curvature does not depend on the choice of the linearly independent vectors X_p, Y_p.

Suggested Exercise 1. (0 points)
Let M be the image of the parametrization $\varphi\colon (0,\infty)\times\mathbb{R} \to \mathbb{R}^3$ given by

$$\varphi(u, v) = (u\cos v, u\sin v, v)$$

and let N be the image of the parametrization $\psi\colon (0,\infty)\times\mathbb{R} \to \mathbb{R}^3$ given by

$$\varphi(u, v) = (u\cos v, u\sin v, \log u).$$

Consider in both M and N the Riemannian metric induced by the Euclidean metric of $\mathbb{R}^3$. Show that the map $f\colon M \to N$ defined by

$$f(\varphi(u, v)) = \psi(u, v)$$

preserves the Gauß curvature but is not a local isometry.

Suggested Exercise 2. (0 points)
Compute the Gauß curvature of:

(a) the sphere S^2 with the standard metric;

(b) the hyperbolic plane H (see Mandatory Exercise 2 on Sheet 9).

Suggested Exercise 3. (0 points)
Show that Ric is the only independet contraction of the curvature tensor, i.e. choosing any other two indices and contracting, one either gets $\pm\,\mathrm{Ric}$ or 0.

Suggested Exercise 4. (0 points)
Let M be a 3-dmensional Riemannian manifold. Show that the curvature tensor is entirely determined by the Ricci tensor.

Suggested Exercise 5. (0 points)
If ∇ is not the Levi-Civita connection can we still define the Ricci curvature tensor Ric? Is it necessarily symmetric?

Suggested Exercise 6. (0 points)
Prove that the Riemann tensor is really a $(1,3)$-tensor.

Suggested Exercise 7. (0 points)
Express the Riemann tensor in local coordinates.

Hand in: Monday 27th June
in the exercise session
in Seminar room 2, MI